

THE δ -VECTORS OF REFLEXIVE POLYTOPES AND OF THE DUAL POLYTOPES

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ABSTRACT. Let $\delta(\mathcal{P})$ be the δ -vector of a reflexive polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d and $\delta(\mathcal{P}^\vee)$ the δ -vector of the dual polytope $\mathcal{P}^\vee \subset \mathbb{R}^d$. In general, $\delta(\mathcal{P}) = \delta(\mathcal{P}^\vee)$ does not hold. In this paper, we give a higher-dimensional construction of a reflexive polytope whose δ -vector equals the δ -vector of the dual polytope. In particular, we consider the case that the reflexive polytope and the dual polytope are unimodularly equivalent.

INTRODUCTION

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope, that is, a convex polytope whose vertices have integer coordinates, of dimension d . Given integers $n = 1, 2, \dots$, we write $i(\mathcal{P}, n)$ for the number of integer points belonging to $n\mathcal{P}$, where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$. In other words,

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^d|, \quad n = 1, 2, \dots$$

In the late 1950s, Ehrhart succeeded in proving that $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$. ([3, Example 35.11]). We call $i(\mathcal{P}, n)$ the *Ehrhart polynomial* of \mathcal{P} .

The generating function of the integral point enumerator, i.e., the formal power series

$$\text{Ehr}_{\mathcal{P}}(t) = 1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)t^n$$

is called the *Ehrhart series* of \mathcal{P} . It is well known that it can be expressed as a rational function of the form

$$\text{Ehr}_{\mathcal{P}}(t) = \frac{\delta_0 + \delta_1 t + \dots + \delta_d t^d}{(1-t)^{d+1}}.$$

The sequence of the coefficients of the polynomial in the numerator

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$$

is called the δ -vector of \mathcal{P} .

The δ -vector has the following properties:

- $\delta_0 = 1$, $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^d| - (d+1)$ and $\delta_d = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^d|$. Hence, $\delta_1 \geq \delta_d$;
- Each δ_i is nonnegative ([10]);
- If $\delta_d \neq 0$, then one has $\delta_1 \leq \delta_i$ for every $1 \leq i \leq d-1$ ([5]).

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We refer the reader to [1, 2, 3, 11, 12, 13] for further informations on Ehrhart polynomials and δ -vectors.

An integral convex polytope is called *reflexive* if the origin of \mathbb{R}^d is a unique integer point belonging to the interior $\mathcal{P} - \partial\mathcal{P}$ of \mathcal{P} and its dual

$$\mathcal{P}^\vee := \{y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in \mathcal{P}\}$$

is also an integral polytope, where $\langle x, y \rangle$ is the usual inner product of \mathbb{R}^d .

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d containing the origin in its interior and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ its δ -vector. It follows from [4] that the following conditions are equivalent:

- \mathcal{P} is reflexive;
- $\delta(\mathcal{P})$ is symmetric, i.e., $\delta_i = \delta_{d-i}$ for every $0 \leq i \leq d$.

Let $\mathcal{P} \subset \mathbb{R}^d$ be a reflexive polytope of dimension d . In general, $\delta(\mathcal{P}) = \delta(\mathcal{P}^\vee)$ does not hold.

It is known from work of Lagarias and Ziegler ([7]) that there are only finitely many reflexive polytopes (up to unimodular equivalence) in each dimension, with one reflexive polytope in dimension one, 16 in dimension two, 4,319 in dimension three, and 473,800,776 in dimension four according to computations by Kreuzer and Skarke ([6]). By computing the number of reflexive polytopes whose δ -vectors equal the δ -vectors of the dual polytopes, we find that there are 4 such reflexive polytopes in dimension two and 327 in dimension three.

It is known that for each $d \geq 2$ there exists a reflexive simplex of dimension d whose δ -vector equals the δ -vector of the dual polytope ([8]). In particular, this reflexive simplex and the dual polytope are unimodularly equivalent. However, other examples of such a reflexive polytope are known very little.

In this paper, we give examples of a reflexive polytope whose δ -vector equals the δ -vector of the dual polytope. In section 1, we give a higher-dimensional construction of a reflexive polytope whose δ -vector equals the δ -vector of the dual polytope (Theorem 1.7). In section 2, we give a new example of a reflexive simplex whose δ -vector equals the δ -vector of the dual polytope (Theorem 2.2).

1. A HIGHER-DIMENSIONAL CONSTRUCTION OF A SPECIAL REFLEXIVE POLYTOPE

Let $\mathbb{Z}^{d \times d}$ denote the set of $d \times d$ integral matrices. Recall that a matrix $A \in \mathbb{Z}^{d \times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral convex polytopes \mathcal{P} and \mathcal{Q} in \mathbb{R}^d of dimension d , we say that \mathcal{P} and \mathcal{Q} are *unimodularly equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and an integral vector w , such that $\mathcal{Q} = f_U(\mathcal{P}) + w$, where f_U is the linear transformation in \mathbb{R}^d defined by U , i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$. In this case we write $\mathcal{P} \cong \mathcal{Q}$. Clearly, if $\mathcal{P} \cong \mathcal{Q}$, then $\delta(\mathcal{P}) = \delta(\mathcal{Q})$. Moreover, if $\mathcal{P} \subset \mathbb{R}^d$, then $\text{Vol}(\mathcal{P})$ denotes the (normalized) *volume* of \mathcal{P} , i.e., $d!$ times the usual euclidean volume of \mathcal{P} .

Example 1.1. Let $\mathcal{P} \subset \mathbb{R}^2$ be the reflexive polytope with the vertices $(1, 0)$, $(-1, 2)$ and $(-1, -1)$. Then the dual polytope \mathcal{P}^\vee has the vertices $(-1, 0)$, $(1, -2)$ and $(1, 1)$. Clearly, $\mathcal{P} \cong \mathcal{P}^\vee$. Therefore $\delta(\mathcal{P}) = \delta(\mathcal{P}^\vee)$.

However, other examples of such a reflexive polytope, i.e., a reflexive polytope and the dual polytope are unimodularly equivalent, are known very little. In this section, we give a higher-dimensional construction of a reflexive polytope whose δ -vector equals the δ -vector of the dual polytope.

For $d \geq 2$ and an integral convex polytope $\mathcal{P} \subset \mathbb{R}^{d-1}$ of dimension $d-1$, we set

$$\begin{aligned}\mathcal{A}(\mathcal{P}) &= \mathcal{P} \times [-1, 1], \\ \mathcal{B}(\mathcal{P}) &= \text{CONV}(\{\mathcal{P} \times \{0\}, (0, 0, \dots, 0, 1), (0, 0, \dots, 0, -1)\}) \subset \mathbb{R}^d, \\ \Gamma(\mathcal{P}) &= \text{CONV}(\{\mathcal{P} \times [-1, 0], (0, 0, \dots, 0, 1)\}) \subset \mathbb{R}^d.\end{aligned}$$

We recall that if \mathcal{P} is reflexive, then $\mathcal{A}(\mathcal{P})$ and $\mathcal{B}(\mathcal{P})$ are also reflexive. Moreover, we have $\mathcal{A}(\mathcal{P})^\vee = \mathcal{B}(\mathcal{P}^\vee)$ and $\mathcal{B}(\mathcal{P})^\vee = \mathcal{A}(\mathcal{P}^\vee)$. $\Gamma(\mathcal{P})$ is an analogy between $\mathcal{A}(\mathcal{P})$ and $\mathcal{B}(\mathcal{P})$.

At first, we show that if \mathcal{P} is reflexive, then $\Gamma(\mathcal{P})$ is a reflexive polytope of dimension d .

Proposition 1.2. *For $d \geq 2$, let $\mathcal{P} \subset \mathbb{R}^{d-1}$ be a reflexive polytope of dimension $d-1$. Then $\Gamma(\mathcal{P})$ is a reflexive polytope of dimension d . Moreover, $\Gamma(\mathcal{P}^\vee) \cong \Gamma(\mathcal{P})^\vee$.*

Before proving Proposition 1.2, we give the following lemma.

Lemma 1.3 ([3, Corollary 35.6]). *Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope dimension d containing the origin in its interior. Then a point $a \in \mathbb{R}^d$ is a vertex of \mathcal{P}^\vee if and only if $\mathcal{H} \cap \mathcal{P}$ is a facet of \mathcal{P} , where \mathcal{H} is the hyperplane*

$$\{x \in \mathbb{R}^d \mid \langle a, x \rangle = 1\}$$

in \mathbb{R}^d .

Now, we prove Proposition 1.2.

Proof of Proposition 1.2. Let $\mathcal{F}_1, \dots, \mathcal{F}_s$ be facets of \mathcal{P} and for $1 \leq i \leq s$, and let \mathcal{H}_i be the hyperplane satisfying $\mathcal{F}_i = \mathcal{P} \cap \mathcal{H}_i$. Then $\Gamma(\mathcal{P})$ has $2s+1$ facets. By Lemma 1.3, we can assume that for $1 \leq i \leq s$,

$$\mathcal{H}_i = \{x \in \mathbb{R}^{d-1} \mid \langle a_i, x \rangle = 1\},$$

where $a_i \in \mathbb{Z}^{d-1}$. Set

$$\mathcal{F}'_i = \begin{cases} \text{CONV}(\{\mathcal{F}_i \times \{0\}, (0, \dots, 0, 1)\}) & i = 1, \dots, s, \\ \mathcal{F}_{i-s} \times [-1, 0] & i = s+1, \dots, 2s, \\ \mathcal{P} \times \{-1\} & i = 2s+1. \end{cases}$$

Then $\mathcal{F}'_1, \dots, \mathcal{F}'_{2s+1}$ are facets of $\Gamma(\mathcal{P})$. For $1 \leq i \leq 2s+1$ let \mathcal{H}'_i be the hyperplane satisfying $\mathcal{F}'_i = \Gamma(\mathcal{P}) \cap \mathcal{H}'_i$. Then

$$\mathcal{H}'_i = \begin{cases} \{x \in \mathbb{R}^d \mid \langle (a_i, 1), x \rangle = 1\} & i = 1, \dots, s, \\ \{x \in \mathbb{R}^d \mid \langle (a_{i-s}, 0), x \rangle = 1\} & i = s+1, \dots, 2s, \\ \{x \in \mathbb{R}^d \mid \langle (0, \dots, 0, -1), x \rangle = 1\} & i = 2s+1. \end{cases}$$

Hence by Lemma 1.3, $\Gamma(\mathcal{P})$ is a reflexive polytope of dimension d .

Moreover, since a_1, \dots, a_s are the vertices of \mathcal{P}^\vee , it clearly follows that $\Gamma(\mathcal{P}^\vee) \cong \Gamma(\mathcal{P})^\vee$. \square

Next, we present a direct formula for the computation of the δ -vector of $\Gamma(\mathcal{P})$ in terms of the δ -vector of \mathcal{P} .

Proposition 1.4. *For $d \geq 2$, let $\mathcal{P} \subset \mathbb{R}^{d-1}$ be an integral convex polytope of dimension $d-1$, and we let $\delta(\mathcal{P}) = (\delta_0(\mathcal{P}), \delta_1(\mathcal{P}), \dots, \delta_{d-1}(\mathcal{P}))$ and $\delta(\Gamma(\mathcal{P})) = (\delta_0(\Gamma(\mathcal{P})), \delta_1(\Gamma(\mathcal{P})), \dots, \delta_d(\Gamma(\mathcal{P})))$ be the δ -vectors of \mathcal{P} and $\Gamma(\mathcal{P})$. Then for $i = 0, 1, \dots, d$, we have*

$$\delta_i(\Gamma(\mathcal{P})) = (i+1)\delta_i(\mathcal{P}) + (d-i+1)\delta_{i-1}(\mathcal{P}),$$

where $\delta_{-1}(\mathcal{P}) = \delta_d(\mathcal{P}) = 0$.

In order to prove Proposition 1.4, we use the following lemmas.

Lemma 1.5. *For $d \geq 2$, let $\mathcal{P} \subset \mathbb{R}^{d-1}$ be an integral convex polytope of dimension $d-1$. Set*

$$\mathcal{Q} = \mathcal{P} \times [0, 1] \subset \mathbb{R}^d,$$

and we let $\delta(\mathcal{P}) = (\delta_0(\mathcal{P}), \delta_1(\mathcal{P}), \dots, \delta_{d-1}(\mathcal{P}))$ and $\delta(\mathcal{Q}) = (\delta_0(\mathcal{Q}), \delta_1(\mathcal{Q}), \dots, \delta_d(\mathcal{Q}))$ be the δ -vectors of \mathcal{P} and \mathcal{Q} . Then for $i = 0, 1, \dots, d$, we have

$$\delta_i(\mathcal{Q}) = (i+1)\delta_i(\mathcal{P}) + (d-i)\delta_{i-1}(\mathcal{P}),$$

where $\delta_{-1}(\mathcal{P}) = \delta_d(\mathcal{P}) = 0$.

Proof. We know $i(\mathcal{Q}, n) = (n+1) \cdot i(\mathcal{P}, n)$. Hence we have

$$\delta_j(\mathcal{Q}) = \sum_{k=0}^j \binom{d+1}{k} (-1)^k (j-k+1) \cdot i(\mathcal{P}, j-k).$$

Since

$$\delta_j(\mathcal{P}) = \sum_{k=0}^j \binom{d}{k} (-1)^k \cdot i(\mathcal{P}, j-k),$$

we obtain

$$(j+1)\delta_j(\mathcal{P}) + (d-j)\delta_{j-1}(\mathcal{P}) = \delta_j(\mathcal{Q}),$$

as desired. \square

Lemma 1.6 ([1, Theorem 2.4]). *For $d \geq 2$, let $\mathcal{P} \subset \mathbb{R}^{d-1}$ be an integral convex polytope of dimension $d-1$. Set*

$$\mathcal{Q} = \text{CONV}(\{\mathcal{P} \times \{0\}, (0, 0, \dots, 0, 1)\}) \subset \mathbb{R}^d.$$

Then we have $\text{Ehr}_{\mathcal{Q}}(t) = \text{Ehr}_{\mathcal{P}}(t)/(1-t)$.

Now, we prove Proposition 1.4.

Proof of Proposition 1.4. We set $\mathcal{Q}_1 = \text{CONV}(\{\mathcal{P} \times \{0\}, (0, \dots, 0, 1)\})$ and $\mathcal{Q}_2 = \mathcal{P} \times [-1, 0]$. Then $\mathcal{Q}_1 \cup \mathcal{Q}_2 = \Gamma(\mathcal{P})$ and $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \mathcal{P} \times \{0\}$. Hence we have

$$\text{Ehr}_{\Gamma(\mathcal{P})}(t) = \text{Ehr}_{\mathcal{Q}_1}(t) + \text{Ehr}_{\mathcal{Q}_2}(t) - \text{Ehr}_{\mathcal{P}}(t).$$

By Lemma 1.6, $(1 - t) \cdot \text{Ehr}_{\mathcal{Q}_1}(t) = \text{Ehr}_{\mathcal{P}}(t)$. Hence we have

$$\text{Ehr}_{\Gamma(\mathcal{P})}(t) = t \cdot \text{Ehr}_{\mathcal{Q}_1}(t) + \text{Ehr}_{\mathcal{Q}_2}(t).$$

Let $\delta(\mathcal{Q}_1) = (\delta_0(\mathcal{Q}_1), \delta_1(\mathcal{Q}_1), \dots, \delta_d(\mathcal{Q}_1))$ and $\delta(\mathcal{Q}_2) = (\delta_0(\mathcal{Q}_2), \delta_1(\mathcal{Q}_2), \dots, \delta_d(\mathcal{Q}_2))$ be the δ -vectors of \mathcal{Q}_1 and \mathcal{Q}_2 . By Lemma 1.5 and Lemma 1.6, we have

$$\delta_i(\mathcal{Q}_1) = \begin{cases} \delta_i(\mathcal{P}) & i = 0, \dots, d-1, \\ 0 & i = d, \end{cases}$$

and for $i = 0, \dots, d$, we have

$$\delta_i(\mathcal{Q}_2) = (i+1)\delta_i(\mathcal{P}) + (d-i)\delta_{i-1}(\mathcal{P}),$$

where $\delta_{-1}(\mathcal{P}) = \delta_d(\mathcal{P}) = 1$. Hence for $i = 0, \dots, d$,

$$\begin{aligned} \delta_i(\Gamma(\mathcal{P})) &= \delta_{i-1}(\mathcal{Q}_1) + \delta_i(\mathcal{Q}_2) \\ &= \delta_{i-1}(\mathcal{P}) + (i+1)\delta_i(\mathcal{P}) + (d-i)\delta_{i-1}(\mathcal{P}) \\ &= (i+1)\delta_i(\mathcal{P}) + (d-i+1)\delta_{i-1}(\mathcal{P}), \end{aligned}$$

as desired. \square

We give a higher-dimensional construction of a reflexive polytope whose δ -vector equals the δ -vector of the dual polytope. By following theorem, we obtain the construction.

Theorem 1.7. *For $d \geq 2$, let \mathcal{P} and $\mathcal{Q} \subset \mathbb{R}^{d-1}$ be integral convex polytopes of dimension $d-1$ such that each of them has at least one interior integer point. Then we have the following properties:*

- (a) $\mathcal{P} \cong \mathcal{Q}$ if and only if $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{Q})$;
- (b) $\delta(\mathcal{P}) = \delta(\mathcal{Q})$ if and only if $\delta(\Gamma(\mathcal{P})) = \delta(\Gamma(\mathcal{Q}))$,

where $\delta(\mathcal{P})$, $\delta(\mathcal{Q})$, $\delta(\Gamma(\mathcal{P}))$ and $\delta(\Gamma(\mathcal{Q}))$ are the δ -vectors of \mathcal{P} , \mathcal{Q} , $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$.

Remark 1.8. For $d \geq 2$, let $\mathcal{P} \subset \mathbb{R}^{d-1}$ be a reflexive polytope of dimension $d-1$. Then by Proposition 1.2 and Theorem 1.7, we have the following properties:

- (a) $\mathcal{P} \cong \mathcal{P}^\vee$ if and only if $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{P})^\vee$;
- (b) $\delta(\mathcal{P}) = \delta(\mathcal{P}^\vee)$ if and only if $\delta(\Gamma(\mathcal{P})) = \delta(\Gamma(\mathcal{P})^\vee)$,

where $\delta(\mathcal{P})$, $\delta(\mathcal{P}^\vee)$, $\delta(\Gamma(\mathcal{P}))$ and $\delta(\Gamma(\mathcal{P})^\vee)$ are the δ -vectors of \mathcal{P} , \mathcal{Q} , $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{P})^\vee$.

Now, we prove Theorem 1.7.

Proof of Theorem 1.7. (a) Clearly, if $\mathcal{P} \cong \mathcal{Q}$, then $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{Q})$. Conversely, suppose that $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{Q})$. We can assume that the origin of \mathbb{R}^{d-1} belongs to the interior of \mathcal{P} and the interior of \mathcal{Q} , and there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$

such that $\Gamma(\mathcal{Q}) = f_U(\Gamma(\mathcal{P}))$, where f_U is the linear transformation in \mathbb{R}^d defined by U . Let v_1, \dots, v_s be the vertices of \mathcal{P} and w_1, \dots, w_s be the vertices of \mathcal{Q} , and let

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1d} \\ u_{21} & u_{22} & \cdots & u_{2d} \\ \vdots & \vdots & & \vdots \\ u_{d1} & u_{d2} & \cdots & u_{dd} \end{pmatrix}.$$

Since $(0, \dots, 0, 1)$ and $(0, \dots, 0, -1)$ belong to $\Gamma(\mathcal{P})$ and since for each $(x_1, \dots, x_d) \in \Gamma(\mathcal{Q})$, we have $-1 \leq x_d \leq 1$, we know $-1 \leq u_{dd} \leq 1$. If $u_{dd} = 0$, then $f_U((v_i, 0))$ and $f_U((v_i, -1))$ have a common d -th coordinate for $1 \leq i \leq s$. Since $\Gamma(\mathcal{Q})$ has just one vertex whose d -th coordinate equals 1 and since the d -th coordinate of $f_U((0, \dots, 0, 1))$ equals 0, there does not exist a vertex v of $\Gamma(\mathcal{P})$ such that $f_U(v) = (0, \dots, 0, 1)$, a contradiction. If $u_{dd} = -1$, then $f_U((0, \dots, 0, 1)) = (0, \dots, 0, -1)$. However, $(0, \dots, 0, 1)$ is a vertex of $\Gamma(\mathcal{P})$ but $(0, \dots, 0, -1)$ is not a vertex of $\Gamma(\mathcal{Q})$, a contradiction. Hence $u_{dd} = 1$. Since $f_U((0, \dots, 0, 1)) = (0, \dots, 0, 1)$, we have

$$U = \begin{pmatrix} & & * \\ & U' & \vdots \\ & & * \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $U' \in \mathbb{Z}^{(d-1) \times (d-1)}$ is a unimodular matrix. Then for each v_i there exists a vertex w_{j_i} of \mathcal{Q} such that $f_U((v_i, 0)) = (w_{j_i}, -1)$ and $f_U((v_i, 1)) = (w_{j_i}, 0)$. Hence for each v_i we have $f_{U'}(v_i) = w_{j_i}$, where $f_{U'}$ is the linear transformation in \mathbb{R}^{d-1} defined by U' . Therefore, $\mathcal{P} \cong \mathcal{Q}$.

(b) If $\delta(\mathcal{P}) = \delta(\mathcal{Q})$, by Proposition 1.4, we have $\delta(\Gamma(\mathcal{P})) = \delta(\Gamma(\mathcal{Q}))$. Suppose that $\delta(\Gamma(\mathcal{P})) = \delta(\Gamma(\mathcal{Q}))$. We set

$$\delta(\mathcal{P}) = (\delta_0(\mathcal{P}), \delta_1(\mathcal{P}), \dots, \delta_{d-1}(\mathcal{P})),$$

$$\delta(\mathcal{Q}) = (\delta_0(\mathcal{Q}), \delta_1(\mathcal{Q}), \dots, \delta_{d-1}(\mathcal{Q})).$$

By Proposition 1.4, for $i = 1, \dots, d-1$, we have

$$(i+1)(\delta_i(\mathcal{P}) - \delta_i(\mathcal{Q})) + (d-i+1)(\delta_{i-1}(\mathcal{P}) - \delta_{i-1}(\mathcal{Q})) = 0$$

Since $\delta_0(\mathcal{P}) = \delta_0(\mathcal{Q})$, for $i = 0, \dots, d-1$, we have $\delta_i(\mathcal{P}) = \delta_i(\mathcal{Q})$. Hence $\delta(\mathcal{P}) = \delta(\mathcal{Q})$. \square

We let $\mathcal{P} \subset \mathbb{R}^2$ be a reflexive polytope of dimension 2. Then the δ -vector of \mathcal{P} equals the δ -vector of \mathcal{P}^\vee if and only if $\mathcal{P} \cong \mathcal{P}^\vee$. However, there exists a reflexive polytope of dimension 3 whose δ -vector equals the δ -vector of the dual polytope such that it and the dual polytope are not unimodularly equivalent. We give an example of such a reflexive polytope.

Example 1.9 ([3, Example 35.11]). Let $\mathcal{P} \subset \mathbb{R}^3$ be the reflexive polytope with the vertices $(-1, 0, 1)$, $(-1, 0, -1)$, $(1, 1, 1)$, $(1, 1, -1)$, $(0, -1, 1)$ and $(0, -1, -1)$. Then \mathcal{P} has 5 facets. Hence \mathcal{P}^\vee has 5 vertices $(0, 0, 1)$, $(0, 0, -1)$, $(2, -1, 0)$, $(-1, 2, 0)$ and $(-1, -1, 0)$. Therefore \mathcal{P} and \mathcal{P}^\vee are not unimodularly equivalent. However, $\delta(\mathcal{P}) = \delta(\mathcal{P}^\vee) = (1, 8, 8, 1)$.

By using Theorem 1.7 and Example 1.9, we obtain the following corollary.

Corollary 1.10. *For each $d \geq 3$, there exists a reflexive polytope of dimension d such that $\delta(\mathcal{P}) = \delta(\mathcal{P}^\vee)$ but \mathcal{P} and \mathcal{P}^\vee are not unimodularly equivalent.*

2. A NEW EXAMPLE OF A REFLEXIVE SIMPLEX

For $d \geq 2$, we let \mathcal{P} be a reflexive polytope of dimension d and \mathcal{Q} the reflexive polytope of dimension $d + 1$ satisfying the assumption of Theorem 1.7. Clearly, \mathcal{Q} is not simplex. We consider a reflexive simplex whose δ -vector equals the δ -vector of the dual polytope.

First, we give an elementary number-theoretic notion.

Definition 2.1. The well-known recursive sequence ([9, A000058]) of pairwise co-prime natural numbers $b_0 := 2$, $b_n := 1 + b_0 \cdots b_{n-1}$ ($n \geq 1$) is called *Sylvester Sequence*. It starts as $b_0 = 2$, $b_1 = 3$, $b_2 = 7$, $b_3 = 43$, $b_4 = 1807$.

For $d \geq 2$, we let \mathcal{P} be a reflexive simplex of dimension d . It is known

$$(d + 1)^{d+1} \leq \text{Vol}(\mathcal{P})\text{Vol}(\mathcal{P}^\vee) \leq (b_d - 1)^2,$$

and if $\text{Vol}(\mathcal{P}) = b_d - 1$, then $\mathcal{P} \cong \mathcal{P}^\vee$ ([8, Theorem C]). Hence if $\mathcal{P} \cong \mathcal{P}^\vee$, we have $\text{Vol}(\mathcal{P}) \leq b_d - 1$.

In this section for $d \geq 3$ we give a new example of a reflexive simplex of dimension d whose δ -vector equals the δ -vector of the dual polytope. In particular the simplex and the dual polytope are unimodularly equivalent and the volume is less than $b_d - 1$. In fact,

Theorem 2.2. *For $d \geq 3$, let \mathcal{P} be the d -dimensional simplex whose vertices $v_i \in \mathbb{R}^d$, $i = 0, 1, \dots, d$, are of the form:*

$$v_i = \begin{cases} -3e_1 - 2 \sum_{i=2}^d e_i & i = 0, \\ e_1 & i = 1, \\ e_1 + 2e_i & i = 2, 3, \\ e_1 + 2b_{i-4}e_i & i = 4, \dots, d, \end{cases}$$

where e_i denotes the i -th unit vector. Then \mathcal{P} is reflexive and we have $\mathcal{P} \cong \mathcal{P}^\vee$, in particular, $\text{Vol}(\mathcal{P}) < b_d - 1$.

In order to prove Theorem 2.2, we use the following lemma.

Lemma 2.3 ([9, A000058]). *For each $n \geq 0$*

$$\frac{1}{b_0} + \frac{1}{b_1} + \cdots + \frac{1}{b_n} = 1 - \frac{1}{b_0 \cdots b_n}.$$

Now, we prove Theorem 2.2.

Proof of Theorem 2.2. First, we show that \mathcal{P} is reflexive. Let $\mathcal{F}_0, \dots, \mathcal{F}_d$ be facets of \mathcal{P} , which are of the form:

$$\mathcal{F}_i = \text{CONV}(\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_d\}) \quad 0 \leq i \leq d,$$

and for $0 \leq i \leq d$, let \mathcal{H}_i be a hyperplane satisfying $\mathcal{F}_i = \mathcal{P} \cap \mathcal{H}_i$. Then

$$\mathcal{H}_i = \begin{cases} \{(x_1, \dots, x_d) \in \mathbb{R}^d | x_1 = 1\} & i = 0, \\ \{(x_1, \dots, x_d) \in \mathbb{R}^d | x_1 - 2x_i = 1\} & i = 2, \dots, d. \end{cases}$$

Also $\mathcal{H}_1 = \{(x_1, \dots, x_d) \in \mathbb{R}^d | \sum_{i=1}^d a_i x_i = 1\}$, where

$$a_i = \begin{cases} -(4b_0 \cdots b_{d-4} - 1) & i = 1, \\ \frac{4b_0 \cdots b_{d-4}}{2} & i = 2, 3, \\ \frac{4b_0 \cdots b_{d-4}}{2b_{i-4}} & i = 4, \dots, d. \end{cases}$$

In fact, $v_0 \in \mathcal{H}_1$ since

$$\begin{aligned} & 3(4b_0 \cdots b_{d-4} - 1) - 4b_0 \cdots b_{d-4} - 4b_0 \cdots b_{d-4} - \frac{4b_0 \cdots b_{d-4}}{b_0} - \dots - \frac{4b_0 \cdots b_{d-4}}{b_{d-4}} \\ &= -3 + 4b_0 \cdots b_{d-4} \left(1 - \left(\frac{1}{b_0} + \frac{1}{b_1} + \dots + \frac{1}{b_{d-4}}\right)\right) \\ &= -3 + 4b_0 \cdots b_{d-4} \frac{1}{b_0 \cdots b_{d-4}} \quad (\text{Lemma 2.3}) \\ &= 1. \end{aligned}$$

Hence since $a_i \in \mathbb{Z}$ ($1 \leq i \leq d$), by Lemma 1.3, \mathcal{P} is reflexive.

Next, we show that $\mathcal{P} \cong \mathcal{P}^\vee$. By Lemma 1.3, we obtain that w_0, \dots, w_d are the vertices of \mathcal{P}^\vee , where

$$w_i = \begin{cases} e_1 & i = 0, \\ (a_1, \dots, a_d) & i = 1, \\ e_1 - 2e_i & i = 2, \dots, d \end{cases}$$

We set a $d \times d$ matrix

$$U = \begin{pmatrix} 1 & 2 & & & & \\ 2 & 2 & 1 & 1 & \cdots & 1 \\ & 1 & -1 & & & \\ & 1 & & -b_0 & & \\ & \vdots & & & \ddots & \\ & 1 & & & & -b_{d-4} \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

where all other terms are zero. Then by Lemma 2.3, we have

$$\begin{aligned}
\det(U) &= \det \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} (-1)^{d-3} b_0 \cdots b_{d-4} + (-1)^{d-2} \sum_{i=0}^{d-4} \frac{b_0 \cdots b_{d-4}}{b_i} \\
&= (-1)^{d-3} b_0 \cdots b_{d-4} + (-1)^{d-2} \sum_{i=0}^{d-4} \frac{b_0 \cdots b_{d-4}}{b_i} \\
&= (-1)^{d-3} b_0 \cdots b_{d-4} \left(1 - \sum_{i=0}^{d-4} \frac{1}{b_i}\right) \\
&= (-1)^{d-3}.
\end{aligned}$$

Hence U is a unimodular matrix. Let f_U be the linear transformation in \mathbb{R}^d defined by U . Then

$$f_U(w_i) = \begin{cases} v_2 & i = 0, \\ v_1 & i = 1, \\ v_0 & i = 2, \\ v_i & i = 3, \dots, d. \end{cases}$$

Hence $\mathcal{P} = f_U(\mathcal{P}^\vee)$. Therefore we have $\mathcal{P} \cong \mathcal{P}^\vee$.

Finally, we show that $\text{Vol}(\mathcal{P}) < b_d - 1$. If $d = 3$, then $\text{Vol}(\mathcal{P}) = 16 < 42 = b_3 - 1$. We assume that $d \geq 4$. Since for each $n \geq 1$, $b_n > b_0 = 2$, for each $n \geq 0$, we have $b_n > 2^n$. Hence since $d \geq 4$ and since $\text{Vol}(\mathcal{P}) = |2^{d-1}(a_1 - 1)| = 2^{d+1}b_0 \cdots b_{d-4}$, we have

$$b_d - 1 = b_0 \cdots b_{d-1} > 2^{3d-6}b_0 \cdots b_{d-4} > 2^{d+1}b_0 \cdots b_{d-4} = \text{Vol}(\mathcal{P}),$$

as desired. \square

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